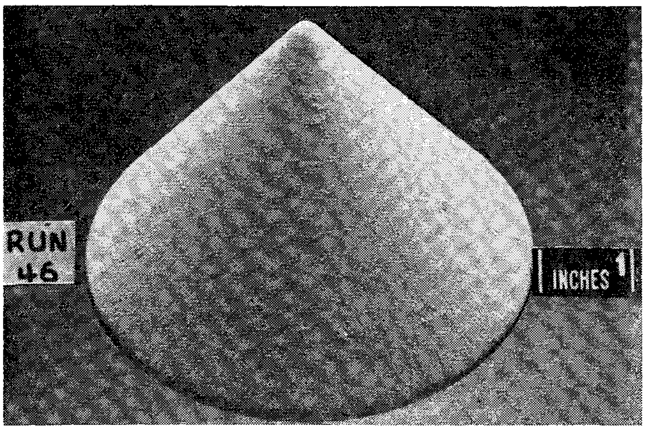

Fig. 3 Teflon (FEP) cone which experienced material creep but which did not ablate during supersonic wind-tunnel experiment carried out by P. R. Nachtsheim (white paint used to accentuate crosshatching). Photograph courtesy NASA Ames Research Center.

should be independent of position on the body or size of the body and should depend only on the material relaxation time. Ballistic range experiments by Canning et al.[2] on ablating Lexan cones of about 0.4-in. slant height led to pattern wavelengths of 0.01–0.02 in., while wind-tunnel tests by Larson and Mateer[3] on ablating Lexan cones with slant heights around 5 in. led to wavelengths of 0.05–0.07 in., an average factor of 4 change in wavelength with more than a factor of 10 change in body size. However, the wavelength change can be accounted for by the different test temperature conditions resulting in a larger value for τ in the wind-tunnel case. Wind-tunnel tests have been run at the Avco Systems Division on ablating Teflon wedges 2 in. in length and the measured wavelengths are about 0.4 in. During the course of the investigation analysis was also made of recovered re-entry flight test vehicles made of Teflon which were 30–40 in. long and wavelengths were found to be in the range 0.3–0.4 in., indicating an essentially unchanged wavelength for a factor of 20 change in body size. The observed wavelengths are consistent in order of magnitude with a relaxation time $\tau \sim 10^{-5}$ sec and a velocity $U \sim 2 \times 10^3$ fps. Data on τ are not readily available but data on G and estimates of μ give the indicated value of τ.

The theoretical result that instability arises for $1 \leqslant M^2 \leqslant 2$ is consistent with the experimental findings that the boundary layer must be turbulent for crosshatching to occur and that the pattern angle closely follows the Mach angle corresponding to the local boundary-layer edge Mach number up to about 2.5–3.0 and then "freezes" out for all higher values.[4] For a turbulent boundary layer at low supersonic Mach numbers the profile is almost uniformly supersonic at a value close to the edge Mach number except immediately adjacent to the wall within the sublayer. On the other hand, at high supersonic Mach numbers in the region of the sublayer edge the Mach number tends to "freeze" out at a value around 2–3.[4] This leads to a characteristic "effective" Mach number within the boundary layer of, say, 1–2. It may be appropriate in considering the three-dimensional pattern to identify this Mach number with that normal to the wavefront.

A Critical Experiment

If the model of the present Note were correct then a nonablating inelastic deformable body could, when placed in supersonic turbulent flow, give rise to crosshatching. P. R. Nachtsheim in cooperation with H. K. Larson of the NASA Ames Research Center kindly agreed to carry out such an experiment in the Ames 3.5-ft Hypersonic Wind Tunnel (see Ref. 3 for a description of the tunnel). The model tested was a 40° half-angle Teflon (FEP) cone of 5 in. base diameter. The material properties were such that at the test conditions of 1673°R stagnation temperature and 1582 psia stagnation pressure and for the 7-sec test time the body did not ablate (either sublime or form a discernible liquid melt) but did exhibit creep. The freestream Mach number was 7.4 and the Mach number at the edge of the boundary layer was 2. The boundary layer itself was turbulent.

When the test was begun nothing was seen to happen for the first few seconds, during which time the model heated up. After this period, however, the material has softened sufficiently and a crosshatch pattern was observed to break out simultaneously over the entire length of the model. When removed from the tunnel the model showed no mass loss to within the measurement accuracy of 0.1 g for the 650 g cone, the pattern angle was consistent with previous measurements (~35°), while the pattern wavelength was around 0.1 in. A photograph of the unablated but crosshatched model is shown in Fig. 3. The blotches seen in the photograph result from the fact that the model was painted white to accentuate the crosshatching.

Concluding Remarks

The results of the nonablating experiment are consistent with those for ablating bodies and lends strong support to the material response model outlined. It may be expected that some details of the model as described will require modification but the deformable relaxing and creeping surface should form the basis for any description of the phenomena.

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Linearly Exact Transverse Displacement Variation

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Nomenclature

A,B = Lamé surface parameters for the undeformed reference surface

$e_{\alpha\alpha},e_{\alpha\beta},e_{\beta\beta}$ = tangential normal and shear strains

$e_{\alpha\gamma},e_{\beta\gamma},e_{\gamma\gamma}$ = transverse shearing strains and transverse normal strain

$\bar{\imath},\bar{\jmath},\bar{k}$ = unit tangent vectors to the α and β coordinate curves of the undeformed surface and the unit normal to that surface

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k_α, k_β = principal curvatures of an undeformed parallel surface
$\bar{n}$ = unit normal to a deformed parallel surface
u,v,w = displacement components of a surface in the $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$ directions, respectively
α,β = principal curvilinear coordinates
γ = transverse coordinate on the undeformed surface

Introduction

THE fundamental problem in two-dimensional linear shell theory still is a rational formulation of the constitutive equations. Implicit in this statement is the problem of an adequate description of the kinematics and kinetics of shell deformation in the transverse coordinate direction.

The derivation presented in the paper defines the linearly exact displacement variation in the transverse coordinate within a shell structure. Corresponding tangential strain variations also are presented as a direct consequence of the displacement variation. The displacement and strain equations consist of two sets of terms. First are those terms that would result from the kinematic constraints of the Love-Kirchhoff hypotheses while the second set of terms appear soley as explicit functions of the transverse normal and shear strains.

The present work differs from and is more general than that presented by Naghdi,[1] Reissner.[2,3] Whereas the referenced works utilize an assumed linear displacement variation in the transverse coordinate and hence are restricted to thin shells, the present work is applicable to thick as well as thin shells. The only limitations of the present work are the applicability of linear theory and that the undeformed shell structure possess parallel lateral bounding surfaces.

The chief advantages that the present work possesses over similar works dealing with either general shell theory[4] or thick shell theory[5] are two in number. First, the displacement variation identifies the contribution of the Kirchhoff-Love constraint terms and the transverse strain terms in an explicit manner. Second, the displacement variation is derived solely on the basis of a study of the kinematics of shell deformation and hence is independent of material characteristics or loading.

Derivation of the Displacement Variation

A. Deformation kinematics

Consider three parallel surfaces before deformation. Let one of the surfaces be termed the reference surface and designated as S_o. Further, let the coordinization on this surface be a principal one with curvilinear coordinates α and β. Assume that the first and second quadratic forms of the reference surfaces are known and given as the following:

$$\mathrm{I} = A_o^2(d\alpha)^2 + B_o^2(d\beta)^2$$

$$\mathrm{II} = -A_o^2 k_{\alpha_o}(d\alpha)^2 - B_o^2 k_{\beta_o}(d\beta)^2$$

Designate the remaining two parallel surfaces as S_i and S_j and assume they are located distance of γ_i and $(\gamma_i + \Delta\gamma_i)$ respectively from the reference surface. Since the surfaces are parallel, then the α-β coordinization of the reference surface is applicable to the parallel surfaces and further, this coordinization also will be a principal one on these surfaces. However, the Lamé surface parameters and the curvatures will differ. Letting the subscripts i and j designate the surface to which the parameters are applicable, then;

$$\begin{aligned} A_i &= A_o(1 + k_{\alpha_o}\gamma_i),\; A_j = A_o[1 + k_{\alpha_o}(\gamma_i + \Delta\gamma_i)] \\ B_i &= B_o(1 + k_{\beta_o}\gamma_i),\; B_j = B_o[1 + k_{\beta_o}(\gamma_i + \Delta\gamma_i)] \\ k_{\alpha_i} &= k_{\alpha_o}/(1 + k_{\alpha_o}\gamma_i),\; k_{\alpha_j} = k_{\alpha_o}/[1 + k_{\alpha_o}(\gamma_i + \Delta\gamma_i)] \\ k_{\beta_i} &= k_{\beta_o}/(1 + k_{\beta_o}\gamma_i),\; k_{\beta_j} = k_{\beta_o}/[1 + k_{\beta_o}(\gamma_i + \Delta\gamma_i)] \end{aligned} \quad (1)$$

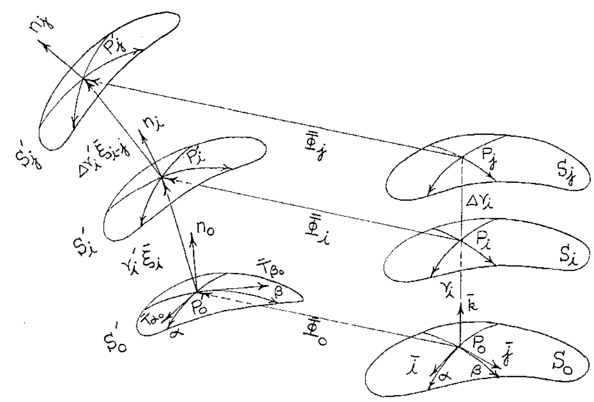

Fig. 1 Reference and parallel surfaces before and after deformation.

An orthonormal right hand system of vectors, $\bar{\imath}$, $\bar{\jmath}$, $\bar{k}$, may be constructed on each of the parallel surfaces such that $\bar{\imath}$ and $\bar{\jmath}$ are the unit tangents to the α and β curvilinear coordinate curves, respectively.

Define the points P_o, P_i, P_j, such that each of the points lie on the surfaces S_o, S_i, and S_j as well as on the common normal to the three surfaces. After deformation, the three parallel surfaces transform to a new set of surfaces designated as S'_o, S'_i, and S'_j. These surfaces in general will no longer be parallel and further, though the same coordinizations, α and β, may be utilized, these coordinizations no longer will be principal ones for the deformed surfaces. The three points, P_o, P_i, P_j, become the points P'_o, P'_i, P'_j which in general will no longer lie along a common normal.

Let the distance between the points P'_o and P'_i be designated as γ'_i and the distance between the points P'_i and P'_j as $\Delta\gamma'_i$. Define $\bar{\xi}_i$ as the unit vector directed from the point P'_o to the point P'_i and $\bar{\xi}_{i-j}$ as the unit vector directed from the point P'_i to P'_j. Further define the three displacement vectors $\bar{\Phi}_o$, $\bar{\Phi}_o$, $\bar{\Phi}_j$, as being the vectors from points P_o, P_i and P_j on the undeformed surface to the corresponding points on the deformed surface.

The three surfaces before and after deformation are shown in Fig. 1. For the two parallel surfaces, S_i and S_j, it is obvious that the following relation is valid:

$$\bar{\Phi}_i + \bar{\xi}_{i-j}\Delta\gamma'_i = \bar{\Phi}_j + \Delta\gamma_i\bar{k} \quad (2)$$

It is equally obvious that for the surfaces S_o and S_i, the following relation holds:

$$\bar{\Phi}_o + \bar{\xi}_i\gamma'_i = \bar{\Phi}_i + \gamma_i\bar{k} \quad (3)$$

Let the components of the displacement vectors $\bar{\Phi}_o$, $\bar{\Phi}_i$, and $\bar{\Phi}_j$ be given as follows:

$$\bar{\Phi}_o = u_o\bar{\imath} + v_o\bar{\jmath} + w_o\bar{k} \quad (4)$$

$$\bar{\Phi}_i = u_i\bar{\imath} + v_i\bar{\jmath} + w_i\bar{k} \quad (5)$$

$$\bar{\Phi}_j = u_j\bar{\imath} + v_j\bar{\jmath} + w_j\bar{k} \quad (6)$$

Thus the displacement components, u_i, v_i, w_i, for any parallel surface could be expressed in terms of the displacement components, u_o, v_o, w_o, of the reference surface if the unit vector $\bar{\xi}_i$ and the distance γ'_i were known. In the following sections, the derivation for these expressions will be given. However, in order to obtain these quantities, first it will be necessary to study the deformation of the two parallel surfaces, S_i and S_j.

B. Adjacent parallel surface deformations and transverse shear strains

Consider now the two parallel surfaces, S_i and S_j and the expression for unit vector $\bar{\xi}_{i-j}$. Substituting Eqs. (5) and (6) into Eqs. (2), the result becomes the following:

$$\bar{\xi}_{i-j}\Delta\gamma'_i = (u_j - u_i)\bar{\imath} + (v_j - v_i)\bar{\jmath} + [(w_j - w_i) + \Delta\gamma_i]\bar{k}$$

However, for a sufficiently small value of $\Delta\gamma_i$, the normal strain, $e_{\gamma\gamma}$, may be assumed to be constant and equal to the value at $\gamma = \gamma_i$. Then,

$$\Delta\gamma'_i = (1 + e_{\gamma\gamma_i})\Delta\gamma$$

and hence the equation for $\bar{\xi}_{i-j}\ \Delta\gamma'_i$ may be written as the following:

$$\bar{\xi}_{i-j}(1 + e_{\gamma\gamma_i})\Delta\gamma_i = (u_j - u_i)\bar{\imath} + (v_j - v_i)\bar{\jmath} + [(w_j - w_i) + \Delta\gamma_i]\bar{k}$$

Dividing by $\Delta\gamma_i$ and passing to the limit as $\Delta\gamma_i \rightarrow 0$, then;

$$\bar{\xi}_{i-j}(1 + e_{\gamma\gamma_i}) = (\partial u_i/\partial\gamma)\bar{\imath} + (\partial v_i/\partial\gamma)\bar{\jmath} + (1 + e_{\gamma\gamma_i})\bar{k} \quad (7)$$

where it is noted that $\partial w_i/\partial\gamma = e_{\gamma\gamma_i}$. Multiplying both sides by $(1 + e_{\gamma\gamma_i})^{-1}$ expanding by the binomial theorem and linearizing, the result becomes

$$\bar{\xi}_{i-j} = (\partial u_i/\partial\gamma)\bar{\imath} + (\partial v_i/\partial\gamma)\bar{\jmath} + \bar{k} \quad (8)$$

The unit tangents, $\bar{T}_{\alpha i}$, $\bar{T}_{\beta i}$, to the α and β coordinate curves of the deformed parallel surface, S_i, and the corresponding unit normal, n_i, are given as;[6]

$$\bar{T}_{\alpha i} = \bar{\imath} + (1/A_i\partial v_i/\partial\alpha - 1/A_iB_i\partial A_i/\partial\beta u_i)\bar{\jmath} + (-k_{\alpha_i}u_i + 1/A_i\partial w_i/\partial\alpha)\bar{k} \quad (9)$$

$$\bar{T}_{\beta i} = (1/B_i\partial u_i/\partial\beta - 1/A_iB_i\partial B_i/\partial\alpha v_i)\bar{\imath} + \bar{\jmath} + (-k_{\beta_i}v_i + 1/B_i\partial w_i/\partial\beta)\bar{k} \quad (10)$$

$$\bar{n}_i = (k_{\alpha_i}u_i - 1/A_i\partial w_i/\partial\alpha)\bar{\imath} + (k_{\beta_i}v_i - 1/B_i\partial w_i/\partial\beta)\bar{\jmath} + \bar{k} \quad (11)$$

The transverse shear strain expression are then the following:

$$e_{\alpha\gamma_i} = \bar{\xi}_{i-j}\cdot\bar{T}_{\alpha i}; \quad e_{\beta\gamma_i} = \bar{\xi}_{i-j}\cdot\bar{T}_{\beta_i}$$

Substituting for these quantities, the result becomes after simplification the known expressions;[7]

$$e_{\alpha\gamma_i} = \partial u_i/\partial\gamma - k_{\alpha_i}u_i + 1/A_i\partial w_i/\partial\alpha \quad (12)$$

$$e_{\beta\gamma_i} = \partial v_i/\partial\gamma - k_{\beta_i}v_i + 1/B_i\partial w_i/\partial\beta \quad (13)$$

Solving for $\partial u_i/\partial\gamma$, $\partial v_i/\partial\gamma$ from Eqs. (12) and (13), substituting the results into Eq. (8) and making use of the expression for the normal, $\bar{n}_i$, the result finally becomes the following:

$$\bar{\xi}_{i-j} = e_{\alpha\gamma_i}\bar{\imath} + e_{\beta\gamma_i}\bar{\jmath} + \bar{n}_j \quad (14)$$

Equation (14) indicates how material points, initially lying on a common normal, vary from surface to surface after deformation. In the absence of transverse shear strains, the material point, P'_j, will be on the normal to adjacent surface S'_i. Similarly, if yet a second adjacent parallel surface, S_k, were constructed so that it lay a distance $\Delta\gamma_j$ above the surface S_j, then the corresponding material point, P_k, after deformation, would lie on the normal to the adjacent deformed surface, S'_j.

The fact that in the absence of transverse shear strains, material points after deformation lie on normals to adjacent surfaces does not insure that the displacement function will be a linear function of the distance normal to the shell reference surface. The normal $\bar{n}_i$, to the deformed parallel surface will vary in direction from one parallel surface to another. Hence its variation must be taken into account in determining the displacement variation.

C. Displacement variation

In the previous section, an expression for $\bar{\xi}_{i-j}$ has been developed. However, this quantity indicates the manner in which material points move after deformation for adjacent surfaces, that is, surfaces spaced an infinitesimal distance apart. In the present section, an expression will be developed for the vector $\bar{\xi}_i$. This quantity governs the motion of material points on surfaces a finite distance γ_i from the reference surface.

Referring to Fig. 1, the displacement vector from the point P'_o to the point P'_i can be thought of as being the sum of displacement vectors between the parallel surfaces located between and including the surfaces S'_o and S'_i.

$$\gamma_i'\bar{\xi}_i = \int_{\gamma=0}^{\gamma=\gamma_i'} \bar{\xi}_{i-j}d\gamma' = \int_{\gamma=0}^{\gamma=\gamma_i} \bar{\xi}_{i-j}(1 + e_{\gamma\gamma})d\gamma$$

But from Eq. (14), after linearization, the above may also be written as

$$\gamma'\xi_i = \left(\int_{\gamma=0}^{\gamma=\gamma_i} e_{\alpha\gamma}d\gamma\right)\bar{\imath} + \left(\int_{\gamma=0}^{\gamma=\gamma_i} e_{\beta\gamma}d\gamma\right) + \left(\int_{\gamma=0}^{\gamma=\gamma_i} (1 + e_{\gamma\gamma})\bar{n}d\gamma\right) \quad (15)$$

Now the expression for $\bar{n}_i$ for the surface S_i has been given by Eq. (11). Differentiating this expression with respect to γ and using the expressions for $e_{\alpha\gamma_i}$, $e_{\beta\gamma_i}$ as stated in Eqs. (12) and (13), the result becomes

$$\partial\bar{n}_i/\partial\gamma = (k_{\alpha_i}e_{\alpha\gamma_i} - 1/A_i\partial e_{\gamma\gamma_i}/\partial\alpha)\bar{\imath} + (k_{\beta_i}e_{\beta\gamma_i} - 1/B_i\partial e_{\gamma\gamma_i}/\partial\beta)\bar{\jmath}$$

Integrating this expression with respect to γ and letting $\bar{n}_o$ designate the normal to the deformed reference surface, the following alternate expression results for $\bar{n}_i$:

$$\bar{n}_i = \bar{n}_o + \left[\int_{\gamma=0}^{\gamma=\gamma_i} k_\alpha e_{\alpha\gamma}d\gamma - \int_{\gamma=0}^{\gamma=\gamma_i} \frac{1}{A}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma\right]\bar{\imath} + \left[\int_{\gamma=0}^{\gamma=\gamma_i} k_\beta e_{\beta\gamma}d\gamma - \int_{\gamma=0}^{\gamma=\gamma_i} \frac{1}{B}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma\right]\bar{\jmath} \quad (16)$$

The expression for $\bar{n}$ given by Eq. (16) may now be substituted into Eq. (15). Since definite integrals are involved, by changing the limits of integration the order of integration may be changed. Linearizing the results and substituting for $\bar{n}_o$ the appropriate form of the expression given by Eq. (11), the result finally becomes,

$$\gamma_i'\bar{\xi}_i = \Bigg[\left(k_{\alpha_o}u_o - \frac{1}{A_o}\frac{\partial w_a}{\partial\alpha}\right)\gamma_i + (1 + \gamma_i k_{\alpha_o}) \times \int_{\gamma=0}^{\gamma=\gamma_i} e_{\alpha\gamma}(1 + \gamma_i k_{\alpha_o})^{-1}d\gamma + \frac{1}{A}\int_{\gamma=0}^{\gamma=\gamma_i}\gamma(1 + \gamma k_{\alpha_o})^{-1} \times \frac{\partial e_{\gamma\gamma}}{\partial\alpha d\gamma}d\gamma - \frac{\gamma_i}{A_o}\int_{\gamma=0}^{\gamma=\gamma_i}(1 + \gamma k_{\alpha_o})^{-1}\frac{\partial e_{\gamma\gamma}}{\partial\alpha d\gamma}d\gamma\Bigg]\bar{\imath} + \Bigg[\left(k_{\beta_o}v_a - \frac{1}{B_o}\frac{\partial w_o}{\partial\beta}\right)\gamma_i + (1 + \gamma_i k_{\beta_o})\int_{\gamma=0}^{\gamma=\gamma_i} e_{\beta\gamma}(1 + \gamma k_{\beta_o})^{-1}d\gamma + \frac{1}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}\gamma(1 + \gamma k_{\beta_o})^{-1}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma - \frac{\gamma_i}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}(1 + \gamma k_{\beta_o})^{-1}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma\Bigg]\bar{\jmath} + \left[\gamma_i + \int_{\gamma=0}^{\gamma=\gamma_i} e_{\gamma\gamma}d\gamma\right]\bar{k} \quad (17)$$

Substituting Eq. (17) into Eq. (3) and using Eqs. (4) and (5), the displacement components on the surface S_i become

$$u_i = u_o + \left(k_{\alpha_o}u_o - \frac{1}{A_o}\frac{\partial w_o}{\partial \alpha}\right)\gamma_i + (1 + \gamma_i k_{\alpha_o}) \times \int_{\gamma=0}^{\gamma=\gamma_i}(1 + k_{\alpha_o}\gamma)^{-1} e_{\alpha\gamma}d\gamma + \frac{1}{A_o}\int_{\gamma=0}^{\gamma=\gamma_i}\gamma(1 + k_{\alpha_o}\gamma)^{-1} \times \frac{\partial e_{\gamma\gamma}}{\partial\alpha d\gamma}d\gamma - \frac{\gamma_i}{A_o}\int_{\gamma=0}^{\gamma=\gamma_i}(1 + k_{\alpha_o}\gamma)^{-1}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma \quad (18)$$

$$v_i = v_o + \left(k_{\beta_o}v_o - \frac{1}{B_o}\frac{\partial w_o}{\partial \beta}\right)\gamma_i + (1 + \gamma_i k_{\beta_o}) \times \int_{\gamma=0}^{\gamma=\gamma_i}(1 + k_{\beta_o}\gamma)^{-1} e_{\beta\gamma}d\gamma + \frac{1}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}\gamma(1 + \gamma k_{\beta_o})^{-1} \times \frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma - \frac{\gamma_i}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}(1 + k_{\beta_o}\gamma)^{-1}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma \quad (19)$$

$$w_i = w_o + \int_{\gamma=0}^{\gamma=\gamma_i} e_{\gamma\gamma}d\gamma \quad (20)$$

D. Tangential strain variation

The tangential strain expressions for a surface are given as the following:[6–8]

$$e_{\alpha\alpha} = 1/A\partial u/\partial\alpha + 1/AB\partial A/\partial\beta v + k_\alpha w$$

$$e_{\alpha\beta} = B/A\partial/\partial\alpha(v/B) + A/B\partial/\partial\beta(u/A)$$

$$e_{\beta\beta} = 1/B\partial v/\partial\beta + 1/AB\partial B/\partial\alpha u + k_\beta w$$

These expressions are applicable to a parallel as well as a reference surface. However, by use of equations (1, 18, 19, 20), the tangential strain expressions for a parallel surface may be expressed in terms of reference surface parameters. Defining the curvature change parameters, κ_α, κ_β, and the torsion τ as the following:

$$\kappa_\alpha = -\frac{1}{A}\frac{\partial}{\partial\alpha}\left(\frac{1}{A}\frac{\partial w}{\partial\alpha}\right) - \frac{1}{AB^2}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\beta} + \frac{k_\beta}{AB}\frac{\partial A}{\partial\beta}v + \frac{1}{A}\frac{\partial}{\partial\alpha}(k_\alpha u)$$

$$\kappa_\beta = -\frac{1}{B}\frac{\partial}{\partial\beta}\left(\frac{1}{B}\frac{\partial w}{\partial\beta}\right) - \frac{1}{A^2B}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\alpha} + \frac{k_\alpha}{AB}\frac{\partial B}{\partial\alpha}u + \frac{1}{B}\frac{\partial}{\partial\beta}(k_\beta v)$$

$$\tau = -\frac{1}{AB}\frac{\partial^2 w}{\partial\alpha\partial\beta} + \frac{1}{A^2B}\frac{\partial A}{\partial\beta}\frac{\partial w}{\partial\alpha} + \frac{1}{AB^2}\frac{\partial B}{\partial\alpha}\frac{\partial w}{\partial\beta} + k_\alpha\frac{A}{B}\frac{\partial}{\partial\beta}\left(\frac{u}{A}\right) + k_\beta\frac{B}{A}\frac{\partial}{\partial\alpha}\left(\frac{v}{B}\right)$$

The result of the discussed substitution becomes the following equations:

$$e_{\alpha\alpha}(\gamma_i) = \frac{1}{(1 + k_{\alpha_o}\gamma_i)}(e_{\alpha\alpha_o} + \kappa_{\alpha_o}\gamma_i) + \frac{1}{A_o(1 + k_{\alpha_o}\gamma_i)}\frac{\partial}{\partial\alpha}\left[(1 + k_{\alpha_o}\gamma_i)\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\alpha\gamma}}{(1 + k_{\alpha_o}\gamma)}d\gamma\right] + \frac{1}{A_oB_o}\frac{(1 + k_{\beta_o}\gamma_i)}{(1 + k_{\alpha_o}\gamma_i)}\frac{\partial A_o}{\partial\beta}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\beta\gamma}}{(1 + k_{\beta_o}\gamma)}d\gamma - \frac{1}{A_o(1 + k_{\alpha_o}\gamma_i)}\frac{\partial}{\partial\alpha}\left[\frac{1}{A_o}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\alpha_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma\right] - \frac{1}{A_oB_o^2}\frac{\partial A_o}{\partial\beta}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\beta_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma + \frac{k_{\alpha_o}}{(1 + k_{\alpha_o}\gamma_i)}\int_{\gamma=0}^{\gamma=\gamma_i}e_{\gamma\gamma}d\gamma \quad (21)$$

$$e_{\alpha\beta}(\gamma_i) = \frac{1}{(1 + 2H_o\gamma_i + K_o\gamma_i^2)}[e_{\alpha\beta_o} + 2\tau_o\gamma_i + (2H_o\tau_o - K_oe_{\alpha\beta_o})\gamma_i^2] + \frac{1}{B_o(1 + k_{\beta_o}\gamma_i)}\frac{\partial}{\partial\alpha}\left[(1 + k_{\alpha_o}\gamma_i) \times \int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\alpha\gamma}}{(1 + k_{\alpha_o}\gamma)}d\gamma\right] - \frac{1}{A_oB_o}\frac{\partial A_o}{\partial\beta}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\alpha\gamma}}{(1 + k_{\alpha_o}\gamma)} \times d\gamma + \frac{1}{A_o(1 + k_{\alpha_o}\gamma_i)}\frac{\partial}{\partial\beta}\left[(1 + k_{\beta_o}\gamma_i)\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\beta\gamma}}{(1 + k_{\beta_o}\gamma)} \times d\gamma\right] - \frac{1}{A_oB_o}\frac{\partial B_o}{\partial\alpha}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\beta\gamma}}{(1 + k_{\beta_o}\gamma)}d\gamma + \frac{1}{A_oB_o^2(1 + k_{\beta_o}\gamma_i)}\frac{\partial B_o}{\partial\alpha}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\beta_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma] + \frac{1}{A_o^2B_o(1 + k_{\alpha_o}\gamma_i)}\frac{\partial A_o}{\partial\beta}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\alpha_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma - \frac{1}{A_o(1 + k_{\alpha_o}\gamma_i)}\frac{\partial}{\partial\beta}\left[\frac{1}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\beta_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma\right] - \frac{1}{B_o(1 + k_{\beta_o}\gamma_i)}\frac{\partial}{\partial\alpha}\left[\frac{1}{A_o}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\alpha_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma\right] \quad (22)$$

$$e_{\beta\beta}(\gamma_i) = \frac{1}{(1 + k_{\beta_o}\gamma_i)}(e_{\beta\beta_o} + \kappa_{\beta_o}\gamma_i) + \frac{1}{B_o(1 + k_{\beta_o}\gamma_i)}\frac{\partial}{\partial\beta}\left[(1 + k_{\beta_o}\gamma_i)\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\beta\gamma}}{(1 + k_{\beta_o}\gamma)}d\gamma\right] + \frac{1}{A_oB_o}\frac{(1 + k_{\alpha_o}\gamma_i)}{(1 + k_{\beta_o}\gamma_i)}\frac{\partial B_o}{\partial\alpha}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{e_{\alpha\gamma}}{(1 + k_{\alpha_o}\gamma)}d\gamma - \frac{1}{B_o(1 + k_{\beta_o}\gamma_i)}\frac{\partial}{\partial\beta}\left[\frac{1}{B_o}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\beta_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\beta}d\gamma\right] - \frac{1}{A_o^2B_o}\frac{\partial B_o}{\partial\alpha}\int_{\gamma=0}^{\gamma=\gamma_i}\frac{(\gamma_i - \gamma)}{(1 + k_{\alpha_o}\gamma)}\frac{\partial e_{\gamma\gamma}}{\partial\alpha}d\gamma + \frac{k_{\beta_o}}{(1 + k_{\beta_o}\gamma_i)}\int_{\gamma=0}^{\gamma=\gamma_i}e_{\gamma\gamma}d\gamma \quad (23)$$

In these equations, H and K are the mean and Gaussian curvatures, respectively.

Conclusions

The displacement component relations (18, 19, 20), and the corresponding strain measures (21, 22, 33) are linearly exact, independent of material characterization and applicable to any continuum bounded laterally by parallel surfaces. Further information regarding the transverse strains cannot be obtained from kinematic considerations. Utilizing the developed strain measures, the continuum compatibility equations degenerate to the two-dimensional reference surface compatibility relations stated in Novozhilov.[8] Hence, insofar as compatible continuum deformations are concerned, the transverse strain variation is arbitrary.

In their present form, the derived displacement and strain expressions are explicitly and implicitly dependent on the transverse coordinate; the latter dependency being manifest in the transverse strain integrals. If three-dimensional shell theory is to be obviated and hence if two-dimensional shell theory constitutive relations are to result, then the implicit dependence must be altered to an explicit one. Such an alteration consists of evaluating the integrals, or, in essence, choosing suitable expressions for the transverse strain variation in the transverse coordinate.

The purpose of this paper has been fulfilled in deriving the linearly exact displacement and strain variation. Yet the obvious goal ultimately is the use of these equations in the construction of a general linear two-dimensional shell theory. Hence, one would be remiss without discussing the evaluation of the integrals.

As derived, the displacement or strain expressions contain the three transverse strains. Other than boundary conditions, no further restrictions exist on these functions. One then could choose arbitrarily and independently a separate functional form for each of the transverse strains and evaluate the integrals. Such an approach would be acceptable but not necessarily satisfactory. Though two-dimensional shell theory is an approximate one, the degree of refinement and applicability of the theory is affected by the approximations. Hence, the more constraints satisfied the better the theory.

The expressions for the displacement or strain components presented in the paper could be used in conjunction with an energy variational theorem. Such an application would relate the transverse strains and the lateral surface loads so that bounds could be established regarding the choice of functional relations for the transverse strains. Ultimately, when the functional form of the transverse strain would be postulated, assurance would be had that the resulting shell theory satisfies the constraints of continuum compatibility and minimum potential energy.

It might be noted that what is being suggested in the previous paragraph as a means of establishing bounds on the functional forms of the transverse strains is utilized in thin shell theory as a means of obtaining the constitutive relations.[9] However, there are important differences between the two applications. In thin shell theory two assumptions are simultaneously introduced into the variational equations. Namely, an assumed linear displacement variation in the transverse coordinate and an assumed transverse stress or strain variation. The fact that the two sets of assumptions are incompatible with each other has been noted by Reissner[10] and also becomes obvious if the assumed strain variations are introduced into the displacement equations presented in the paper. Thus the thin shell application of the variational theorem when used in conjunction with an assumed linear displacement variation and an assumed transverse strain or stress variation leads to a shell theory which satisfies the minimum energy constraint but violates that of continuum compatibility. However, for sufficiently thin shells, the contradiction usually results in small errors.[11,12]

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Surface Structure of Ammonium Perchlorate Composite Propellants

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THE surface structure of a burning solid propellant is an important aspect to be considered when attempting to mechanistically or mathematically model the propellant combustion process. The vast majority of models proposed for steady and nonsteady combustion of solid propellants treat the burning surface as a planar, dry, homogeneous entity with one simple Arrhenius expression describing the chemical reaction of the entire regressing surface. For composite propellants, this simplification does not fully describe the combustion. One justification for such simplification is that the complexity of the model can be reduced to a more tractable mathematical representation. Another reason is that detailed information defining the structure of the surface is not available to construct accurately a more physically realistic model. Because of the latter reason, an experimental investigation was undertaken to understand better the physical nature of the surface and determine, if possible, the extent of heterogeneous or subsurface reactions. In the study, burning propellant samples were extinguished and then examined by using a scanning electron microscope (SEM).

The test propellants consisted of ammonium perchlorate (AP) and either polyurethane (PU) or carboxy-terminated polybutadiene (CTPB) binder. The propellants were formulated with either a unimodal oxidizer particle size distribu-

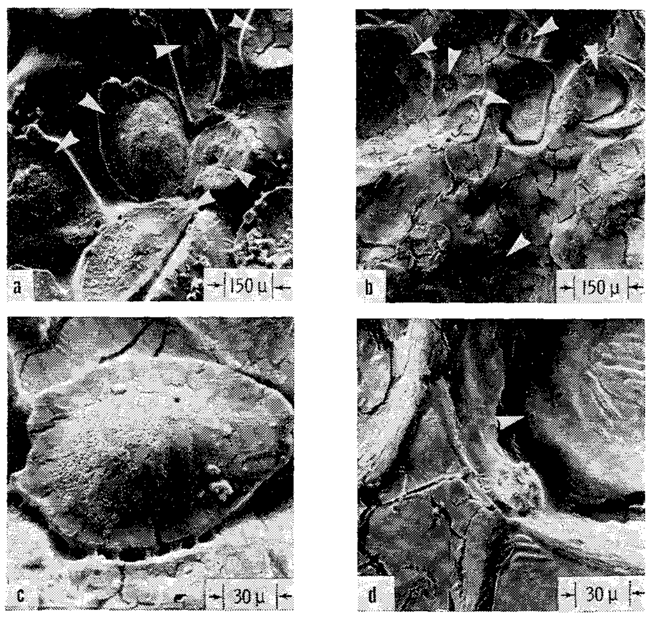

Fig. 1 Scanning electron microscope (SEM) micrographs of an ammonium perchlorate-polyurethane propellant quenched from (a,c) 100 psia and (b,d) 800 psia. The white arrows indicate the ammonium perchlorate particles. Note the different positions of these particles relative to the binder for the two cases.

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